

# On the Separability of Targets Using Binary Proximity Sensors<sup>\*</sup>

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## Abstract

We consider the problem where a network of sensors has to detect the presence of targets at any of  $n$  possible locations in a finite region. All such locations may not be occupied by a target. The data from sensors is fused to determine the set of locations that have targets. We term this the separability problem. In this paper, we address the separability of an asymptotically large number of static target locations by using binary proximity sensors. Two models for target locations are considered: (i) when target locations lie on a uniformly spaced grid; and, (ii) when target locations are i.i.d. uniformly distributed in the area. Sensor locations are i.i.d. uniformly distributed in the same finite region, independent of target locations. We derive conditions on the sensing radius and the number of sensors required to achieve separability. Order-optimal scaling laws, on the number of sensors as a function of the number of target locations, for two types of separability requirements are derived. The robustness or security aspects of the above problem is also addressed. It is shown that in the presence of adversarial sensors, which toggle their sensed reading and inject binary noise, the scaling laws for separability remain unaffected.

## I. INTRODUCTION

Motivated by applications in cognitive radio, and in target sensing situations like wildlife monitoring or land mine detection, we define and develop the separability problem. An important requirement in cognitive radio systems is the detection of white spaces—the regions where the primary radio transmitters are not active. Consider the following white space detection problem considered in [1]. In a region of interest, there are  $n$  possible locations where these primary transmitters could be present. It is reasonable to assume that each of these  $n$  points may contain at most one radio transmitter. To detect whitespace, i.e. the area in where there is no radio reception, a set of radio receivers are deployed randomly and each receiver can determine the existence of a radio signal of strength above a specified threshold. The location of the primary transmitters, and hence the available white space, is to be determined using the binary output of the receivers.

As a second example, consider estimation of the population of rare wildlife in a reserve forest. There are locations in these forests that an animal is expected to visit e.g., watering hole or a salt lick. If the animal is solitary, e.g., tigers or leopards, then at most one of them will be present at any given time at any of these locations. Sensors can be placed to sense the presence or absence of an animal at these sites and the output from the sensors can be used to estimate the population. Such a technique was employed to estimate the tiger population in the Nagarhole reserve forest in India [2] where the forest was overlaid with an approximate grid and sensors were suitably placed to sense the presence of tigers in these sites.

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A third example is of land-mine detection. It is not unreasonable to assume that, say,  $n$  mines, have been randomly placed in an area. Some of these are inert and others active. It is of interest to detecting the location of the active mines using sensors that can determine the presence of an active mine in their coverage range.

The preceding examples motivate the *separability* problem, which is defined next. A finite region of interest, say  $\mathcal{I}$ , has  $n$  points that are called *target locations*. Each of these  $n$  points contains at most one target. An ideal binary proximity sensor of sensing radius  $r(n)$  outputs a ‘1’ if one or more targets are present within its sensing radius  $r(n)$  and outputs ‘0’ otherwise.  $m(n)$  ideal binary proximity sensors are randomly deployed in  $\mathcal{I}$ . The random location of the sensors models the lack of precise control during sensor-deployment but the random realization is assumed known. The objective is to find the *target configuration*—identify the set of target locations that contain a target—using the outputs of these  $m(n)$  sensors. We determine order-optimal conditions on  $r(n)$  and  $m(n)$  to determine the target configuration. This is a significant generalization of the definition of separability described in [3]. In this paper we study several variations of the separability problem for the two following models of target locations.

- 1) *Targets on grid*: where the target locations are on a uniform grid that is overlaid on  $\mathcal{I}$ .
- 2) *Random targets*: where the target locations are i.i.d. realizations of a uniform random variable over  $\mathcal{I}$ .

Clearly, the sensing radius of the binary proximity sensors determines the quality of the separation that is achieved—a large sensing radius lowers the resolution while a small sensing radius requires a larger number of sensors. Thus the sensing radius is a design parameter to be chosen suitably.

We are now ready to state the objective of this work—determine  $(r(n), m(n))$  the sensing radius of each sensor and the number of sensors that are randomly deployed to achieve separability of the  $n$  target locations. For each of the target location models, we seek to find  $r(n)$  and  $m(n)$  for the following two performance criteria.

- 1) *Full separability* where the configuration of all the  $n$  target locations are to be identified correctly. Our results are asymptotic (in  $n$ ) and have the form

$$\Pr(\text{all } 2^n \text{ target configurations can be identified}) \rightarrow 1.$$

- 2) *Partial separability* where the configuration of at least a fraction  $\alpha$ ,  $0 < \alpha < 1$ , of the locations is to be determined correctly with probability at least  $\beta$ ,  $0 < \beta < 1$ , i.e.

$$\Pr(\text{configuration at } \geq \alpha n \text{ target locations are correctly identified}) \geq \beta.$$

#### A. Previous Work

Localization of a source or a target is probably the closest class of problems to separability. This is a very old problem and the literature is replete with source and target localization using a variety of measurement models. See [4] for an excellent survey of localization problems in sensor networks. While a large part of the localization literature considers measurement models like range, angle-of-arrival, etc, binary proximity sensors have also been used in several localization problems e.g., [5], [6]. More recently, binary proximity sensors have also been used in target tracking, e.g., [7], [8]. Another problem closely related to separability is the counting problem—count the number of targets in a finite sensing area [9]. In [9], the counting problem has been studied with sensors that can output the number of distinct targets they can sense, i.e., the output is not binary. We will see below that separability is distinct from both of these.

The notion of separability was introduced in [3] where the following problem was studied. A single target is located at one of two possible locations, say  $t_1$  and  $t_2$ . Binary proximity sensors, possibly non ideal, are deployed in  $\mathbb{R}^2$  according to a spatially homogeneous Poisson process of density  $\lambda$ . The separability problem, identifying which of  $t_1$  and  $t_2$  contains the target, was formulated as a binary hypothesis testing problem and fundamental bounds on the decoding error was obtained using information theoretic techniques. They also consider the case where the sensor output is from an alphabet  $\mathcal{Y}$ . The difference between separability and localization is now apparent—separability is a disambiguation problem while localization is an estimation problem. In [3] it is assumed that the target is present in exactly one of two possible locations; we generalize and consider the case where upto one target can be present at each of  $n$  locations. Thus our disambiguation is between the  $2^n$  possibilities, akin to decoding.

Much of our techniques and results will be closely related to results in coverage problems. It may be noted that building on coverage problems outlined in [10], there has been a significant amount of work on coverage in sensor networks, e.g. see [11], [12]. The primary interest in this line of research is to use random shapes (sensor coverage areas) and cover any subset of  $\mathbb{R}^d$  or a measurable fraction of the subset. Infer that the separability problem reduces to the coverage of a countable number of points with extra restrictions, we will compare our results to analogous results from coverage analysis.

### B. Organization of the Paper and Summary of Results

The rest of the paper is organized as follows. The system model and relevant mathematical results are described in Section II. The main results, i.e., the scaling laws for critical  $r(n)$  and the corresponding  $m(n)$  for the two target models (Theorem 1 and 2) are described in Section III. In Subsection III-A we consider the targets-on-grid model and randomly realized target locations are described in Subsection III-B. For pedagogical convenience, Section III will deal with separability on  $\mathcal{I} = [0, 1]$  and the two dimensional extension is described in Theorem 3 in Section IV.

For secure settings, it is also desirable to have some form of robustness against adversarial sensors; an adversarial sensor can mislead the decision process by injecting binary noise, that toggles its actual reading. This form of adversarial sensing is discussed in Section V where we assume that there is a known upper bound on the fraction of sensors that are adversarial. We will argue in Theorem 4 that majority logic can be used and the order of  $r(n)$  and  $m(n)$  does not change. Finally, conclusions are presented in Section VI.

## II. SYSTEM MODEL AND MATHEMATICAL PRELIMINARIES

In this section, we describe the system model and relevant notation. This is followed by some known mathematical results which will be used in the subsequent sections.

### A. System model

The sensor field is a finite interval  $\mathcal{I}$ ; without loss of generality we assume that,  $\mathcal{I} = [0, 1]$ .  $\mathcal{T}$  is the set of  $n$  ( $n < \infty$ ) distinct points in  $\mathcal{I}$  that are the target locations. Two models for  $\mathcal{T}$  will be used in this work. In the targets-on-grid model, the target locations ( $\mathcal{T}_g$ ) are on a finite grid, i.e.,

$$\mathcal{T}_g := \left\{ \frac{1}{2n}, \frac{3}{2n}, \dots, \frac{(2n-1)}{2n} \right\}.$$

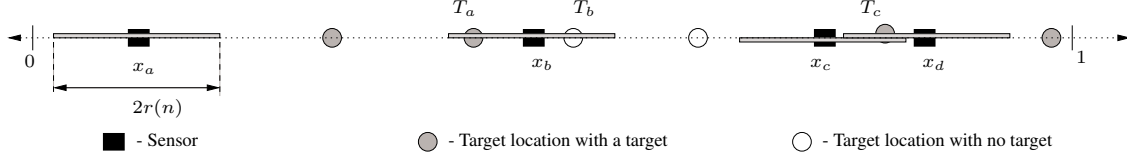


Fig. 1. Illustrating conditions on identifiability of a target locations. Target location  $T_c$  is identifiable. Target locations  $T_a$  and  $T_b$  are not identifiable.

In the random-targets model, the target locations ( $\mathcal{T}_{\text{rnd}}$ ) are a realization of  $n$  i.i.d. random variables uniformly distributed in  $\mathcal{I}$ . They will be represented using the ordered target locations as below.

$$\mathcal{T}_{\text{rnd}} := \{T_{(1)}, T_{(2)}, \dots, T_{(n)}\}$$

Here  $\{T_{(i)}, n \in \mathbb{N}\}$  is the  $i$ -th order statistic of  $n$  i.i.d.  $\text{Uniform}[0, 1]$  random variables. We reiterate that all target locations in  $\mathcal{T}$  need not be occupied by targets.

Recall that, an ideal binary proximity sensor at location  $x$  with sensing radius  $r(n)$  outputs a 1 if and only if there exists at least one target in  $(x - r(n), x + r(n))$ . The locations of the set of  $m(n)$  sensors is denoted by  $\{X_1, X_2, \dots, X_{m(n)}\}$ , where  $X_i$  are i.i.d. uniformly distributed in  $\mathcal{I}$ . Throughout the paper, we assume all sensors to be ideal binary proximity sensors. To detect the possible presence of targets in  $\mathcal{T}$ ,  $m(n)$  sensors are randomly deployed in  $\mathcal{I}$ . Each sensor has a sensing radius of  $r(n)$  i.e., for a sensor at location  $x$ , the sensing region is

$$\mathcal{R}(x, r(n)) = \{y : y \in \mathcal{I} \text{ and } |y - x| < r(n)\}.$$

The sensing radius  $r(n)$  will be treated as a *design parameter*.

The data recording model of the sensors is as follows. A sensor at  $x$  outputs a logical 1 if it detects at least one target in  $\mathcal{R}(x, r(n))$ . We will see from the following argument that target location  $T_i$  is unambiguously identifiable by a sensor if and only if the sensor detects  $T_i$  and no other  $T_j, j \neq i$ . Since we assume that targets can be present only at the target locations in  $\mathcal{T}$ , the following cases prove the above claim.

- 1) For a sensor at  $x$ , if  $T_i \notin \mathcal{R}(x, r(n)) \forall i \in \{1, \dots, n\}$ , then it outputs a logical '0' irrespective of the target configuration and the sensor observation is not useful. The sensor at  $x_a$  in Fig. 1 illustrates this condition.
- 2) For a sensor at  $x$ , and some  $i$  and  $j$ ,  $1 \leq i < j \leq n$ , let  $T_i \in \mathcal{R}(x, r(n))$  and  $T_j \in \mathcal{R}(x, r(n))$ . If at least one of  $T_i$  or  $T_j$  has a target then the sensor at  $x$  will output a '1'. However, this sensor's observation cannot be used to distinguish any configuration of  $T_i$  and  $T_j$  with at least one target. The sensor at  $x_b$  and target locations  $T_a$  and  $T_b$  in Fig. 1 illustrate this condition.
- 3) Let three consecutive target locations  $\{T_{i-1}, T_i, T_{i+1}\}$  be such that  $|T_i - T_{i-1}| < r(n)$  and  $|T_i - T_{i+1}| < r(n)$ , then all sensors that cover  $T_i$  also cover either  $T_{i-1}$  or  $T_{i+1}$ . If there is a target at both  $T_{i-1}$  and  $T_{i+1}$ , then the presence or absence of a target at  $T_i$  cannot be distinguished by any set of sensors.

Thus, a sensor at  $x$  can be used to determine the target configuration at  $T_i$  if and only if  $T_i \in \mathcal{R}(x, r(n))$  and  $T_j \notin \mathcal{R}(x, r(n)) \forall j \neq i$ . This leads us to the following definition. We say that target location  $T_i$  is *identifiable* if there is at least one sensor at  $x \in \mathcal{I}$  such that  $T_i \in \mathcal{R}(x, r(n))$  and  $T_j \notin \mathcal{R}(x, r(n)), \forall j \neq i$ . The target location at  $T_c$  in Fig. 1, is covered

by sensors at  $x_c$  and  $x_d$ , and is identifiable. Thus full separability is equivalent to having  $n$  identifiable targets.

In this paper we seek two types of separability results. In *full separability*, the objective is to determine the asymptotic  $r(n)$  and  $m(n)$  for which every possible target configuration is separated with high probability. In other words, find  $r(n)$  and  $m(n)$  that will, with high probability, identify every target location. The second set of results determine the  $r(n)$  and  $m(n)$  to achieve *partial separability*, i.e., we determine these quantities for which at least  $\alpha n$ ,  $0 < \alpha < 1$ , target locations are identifiable with a probability at least  $\beta$ ,  $0 < \beta < 1$ .

To make this paper self-contained, we next present some mathematical results, some of which are known in the literature, that will be used in our analysis.

## B. Mathematical preliminaries

First, a note on symbols. The set of reals and naturals are denoted by  $\mathbb{R}$  and  $\mathbb{N}$  respectively. We have already used the symbol  $\Pr$  for probability of an event; it is assumed that there is a common  $(\Omega, \mathcal{F}, \Pr)$  structure for defining all the events in this work.

The order notation is well known but we recapitulate them here for completeness. For positive sequences  $f(n)$  and  $h(n)$  we say that  $f(n) = \Theta(h(n))$  if there are non-zero positive constants  $0 < a_1 < a_2$  and a corresponding  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a_1 h(n) \leq f(n) \leq a_2 h(n)$ . Similarly, we say that  $f(n) = \omega(h(n))$  if  $\lim_{n \rightarrow \infty} f(n)/h(n) = \infty$ .

The following lemma bounds the asymptotic behavior of  $(1 - \theta)^m$ .

*Lemma 1:* For constant  $\theta$ ,  $0 < \theta < 1$  and any positive integer  $m$ ,

$$\exp\left(-\frac{m\theta}{1-\theta}\right) < (1-\theta)^m < \exp(-m\theta). \quad (1)$$

This implies that  $(1 - \theta)^m \rightarrow 0$  if and only if  $\exp(-m\theta) \rightarrow 0$ .

*Proof:* For  $0 < \theta < 1$ ,  $(1 - \theta) < \exp(-\theta)$ , and thus the upper bound follows. The lower bound is obtained from  $\exp(x) > 1 + x$ , for  $x > 0$ , using  $x = \frac{\theta}{1-\theta}$ . ■

The following results from order statistics are adapted from [13, pg. 134]. Let  $\{U_i, 1 \leq i \leq n\}$  be i.i.d. Uniform[0, 1] random variables and  $U_{(i)}, 1 \leq i \leq n$  be their order statistics, i.e.,  $U_{(1)} \leq U_{(2)} \leq \dots \leq U_{(n)}$ . Let  $U_{(0)} := 0$ . Define the spacing variables as follows:  $V_1 = U_{(1)}$ ,  $V_i = U_{(i)} - U_{(i-1)}$ ,  $2 \leq i \leq n$ , and  $V_{n+1} = 1 - U_{(n)} = 1 - \sum_{i=1}^n V_i$ . The joint probability density function of  $\{V_i, 1 \leq i \leq n\}$ ,

$$f_{V_1, \dots, V_n}(v_1, \dots, v_n) = \begin{cases} n! & \text{for } v_i \geq 0 \text{ and } \sum_{i=1}^n v_i \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\Pr(V_1 > v_1, \dots, V_n > v_n) = \begin{cases} (1 - v_1 - \dots - v_n)^n, & \text{if } \sum_{i=1}^n v_i \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Since the probability density function  $f_{V_1, \dots, V_n}(v_1, \dots, v_n)$  is symmetric, the distribution of any  $k$  spacings,  $1 \leq k \leq n$ , has the same distribution as that of the first  $k$  spacings, i.e., of  $V_1, \dots, V_k$ . This is obtained by setting the  $v_i = 0$  for the other  $(n - k)$  spacings in (2). Thus, for any  $k < n$  and  $1 \leq n_1 < n_2 < \dots < n_k \leq n$ ,

$$\Pr(V_{n_1} > v_1, \dots, V_{n_k} > v_k) = \begin{cases} (1 - v_1 - \dots - v_k)^n & \text{if } v_i > 0 \text{ and } \sum_{i=1}^k v_i \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Next, we derive a version of the Markov inequality on the sum of Bernoulli random variables. Let  $\{B_n\}$ ,  $n > 0$ , be a sequence of i.i.d. Bernoulli random variables with parameter  $p$  and let  $S_n := \sum_{i=1}^n B_i$ . From Markov inequality on  $n - S_n$ , we have

$$\begin{aligned}\Pr((n - S_n) \geq (1 - \alpha)n) &\leq \frac{(1 - p)n}{(1 - \alpha)n} \\ \Pr(S_n \leq n\alpha) &\leq \frac{1 - p}{1 - \alpha} \\ \Pr(S_n > n\alpha) &= 1 - \Pr(S_n \leq n\alpha) \geq \frac{p - \alpha}{1 - \alpha}.\end{aligned}\tag{4}$$

Observe that this bound is uniform in  $n$ .

We now summarize some results for the coupon collector problem [14]. Recall that in the coupon collector problem, there are  $n$  distinct coupons in a bag and coupons are sampled with replacement. The quantity of interest is the minimum number of samples so that each coupon is sampled at least once. Let  $E_i$  indicate that coupon  $i$  has not been sampled in  $m$  draws. The following equality relations asymptotically hold [14].

- 1) For any constant  $c > 0$ , if  $m = n(\log n - c)$  then  $\lim_{n \rightarrow \infty} \Pr(\sum_{i=1}^n E_i \geq 1) = 1 - \exp(-\exp(c))$ . If instead of  $c$ , we use any  $c_n \rightarrow \infty$ , then  $\lim_{n \rightarrow \infty} \exp(-\exp(c_n)) = 0$  and  $\Pr(\sum_{i=1}^n E_i \geq 1) \rightarrow 1$ .
- 2) If  $m = n(\log n + c)$ , then  $\lim_{n \rightarrow \infty} \Pr(\sum_{i=1}^n E_i \geq 1) = 1 - \exp(-\exp(-c))$ . The following hold.
  - a) For any real positive constant  $c$ ,  $\lim_{n \rightarrow \infty} \Pr(\sum_{i=1}^n E_i \geq 1) < 1$ .
  - b) If instead of  $c$ , we use any  $c_n \rightarrow \infty$ , then  $\lim_{n \rightarrow \infty} \exp(-\exp(-c_n)) = 1$ . Thus  $\Pr(\sum_{i=1}^n E_i \geq 1) \rightarrow 0$ .

Thus, if  $m$  is the number of samples needed to sample all of the  $n$  coupons, then  $m \geq n(\log n + c_n)$  for any  $c_n \rightarrow \infty$ . Observe that to ensure that every coupon has been drawn  $m/n$  is logarithmic.

The main results are presented in the next section.

### III. SCALING LAWS FOR SEPARABILITY

We first consider the targets-on-grid model and then consider the random targets model.

#### A. Separability of target locations on a grid

For notational convenience, the targets in  $\mathcal{T}_g$  will be numbered  $1, 2, \dots, n$  from the left. Recall that in  $\mathcal{T}_g$ , the  $i$ -th target location  $T_i = (2i - 1)/(2n)$ ,  $1 \leq i \leq n$ . Theorem 1 presents the results of this subsection.

*Theorem 1 (Separability of targets-on-grid):* For the sensing region  $\mathcal{I} = [0, 1]$ , and target locations  $\mathcal{T}_g$ , when  $m(n)$  sensors are deployed uniformly in  $\mathcal{I}$ ,

- 1)  $0 < r(n) < (1/n)$  is necessary for separability.
- 2) Let  $r(n) = a/2n$ , or  $r(n) = (2 - a)/2n$ , for  $0 < a \leq 1$ , then the following are true.
  - a) If  $m(n) \geq (n/a)(\log(n/a) + c_n)$ , for any  $c_n \rightarrow \infty$ , then  $\Pr(\text{all target configurations are separable}) \rightarrow 1$ .
  - b) If  $m(n) \leq (n/a)(\log(n/a) - c_n)$ , for any  $c_n \rightarrow \infty$ , then  $\Pr(\text{all target configurations are separable}) \rightarrow 0$ .

3) Let  $r(n) = a/2n$ , or  $r(n) = (2-a)/2n$ , for  $0 < a \leq 1$ . Given  $0 < \alpha < 1$ , and  $0 < \beta < 1$ , the following are true.

- a) If  $m(n) \geq (n/a) \log \left( \frac{1}{(1-\alpha)(1-\beta)} \right)$  then  $\Pr(\text{at least } \alpha n \text{ targets are separable}) > \beta$ .
- b) If  $m(n) < (n/a - 1) \log \left( \frac{1}{(1-\alpha)\beta} \right)$  then  $\Pr(\text{at least } \alpha n \text{ targets are separable}) < \beta$ .

*Proof:* We first prove statement 1.

- If  $r(n) > 1/n$ , then all sensors have at least two target locations in their sensing region. From our discussion in Section II-A, it follows that no target is identifiable.
- If  $r(n) = 1/n$ , then only sensors placed at  $T_i$  sense exactly one target while any sensor at other locations senses two target locations. In a random sensor deployment, having sensors at the target locations  $\mathcal{T}_g$  has zero probability, thus targets are not separable with probability 1.

This proves statement 1 that  $0 < r(n) < 1/n$  is necessary for separability. Next we prove statement 2.

First let  $r(n) = a/2n$ , with  $0 < a \leq 1$ . From the uniform distribution of the sensors, a sensor covers target location  $i$  with probability  $a/n$ . For  $r(n) \leq 1/2n$ , if a target location is covered, then it is identifiable. Thus with the  $n$  target locations as coupons and the  $m(n)$  sensors as draws, this is analogous to the coupon collector problem. For full separability we need all the target locations to be covered by at least one sensor; hence statement 2 follows.

Next consider  $r(n) = (2-a)/2n$  with  $0 < a \leq 1$ . For each  $T_i$ , any sensor in the interval  $\mathcal{I}_i := (T_i - (a/2n), T_i + (a/2n))$  covers only  $T_i$  while a sensor elsewhere that covers  $T_i$  will also cover  $T_{i-1}$  or  $T_{i+1}$ . From our discussion in Section II-A,  $T_i$  is identifiable if and only there is at least one sensor in  $\mathcal{I}_i$ . The probability that a uniformly deployed sensor node falls in  $\mathcal{I}_i$  is  $a/n$ , which is the same as that for  $r(n) = a/2n$  for  $0 < a \leq 1$ . The rest of the proof for  $r(n) = (2-a)/2n$  follows analogous to the case of  $r(n) = a/2n$ . This completes the proof of statement 2.

Before we prove statement 3, we first derive upper and lower bounds on the probability of having at least  $\alpha n$  identifiable target locations. First, let  $r(n) = a/2n$ ,  $0 < a \leq 1$ . Recall that for this  $r(n)$ , a target is identifiable if and only if it is covered. Let  $\mathcal{E}_i$  be the indicator of the event that  $T_i$  is not covered. For partial separability, we require  $\Pr(\sum_{i=1}^n (1 - \mathcal{E}_i) \geq \alpha n) \geq \beta$ . Using (4) and the Markov inequality on  $(1 - \mathcal{E}_i)$ , we have the following lower and upper bounds respectively. For notational convenience, we will use  $m$  instead of  $m(n)$  for the rest of this proof.

$$\frac{1 - (1 - (a/n))^m - \alpha}{1 - \alpha} \leq \Pr \left( \sum_{i=1}^n (1 - \mathcal{E}_i) \geq \alpha n \right) \leq \frac{1 - (1 - (a/n))^m}{\alpha}. \quad (5)$$

Now, we prove statement 3a. Let  $m(n)$  be chosen such that  $m \geq \left( \frac{n}{a} \right) \log \left( \frac{1}{(1-\alpha)(1-\beta)} \right)$ . By appropriate manipulations and (1), the lower bound in (5) is  $> \beta$  as shown below.

$$(1 - (a/n))^m < \exp \left( -\frac{am}{n} \right) \leq (1 - \alpha)(1 - \beta) \iff 1 - \frac{(1 - (a/n))^m}{1 - \alpha} > \beta.$$

This completes the proof of statement 3a. Next, we prove statement 3b using (5). Let  $m(n)$  be chosen such that  $m < \left( \frac{n}{a} - 1 \right) \log \left( \frac{1}{1-\alpha\beta} \right)$ . By algebraic manipulations and using (1), we see that the upper bound from (5) is  $< \beta$  as shown below.

$$(1 - (a/n))^m > \exp \left( -\frac{m}{\frac{n}{a} - 1} \right) > 1 - \alpha\beta \iff \frac{1 - (1 - (a/n))^m}{\alpha} < \beta.$$

This completes the proof of statement 3 for  $r(n) = a/2n$ , with  $0 < a \leq 1$ . If  $r(n) = (2-a)/2n$ , for  $0 < a \leq 1$ , then by an argument identical to the proof of statement 2, the conditions on  $m(n)$  are identical to that with  $r(n) = a/2n$ . ■

*Remark 1:* If  $r(n) = 1/2n$  then full coverage of  $\mathcal{I}$ , as defined in [10], is a sufficient condition for full separability. Note that in the coverage analysis in [10], sensors are distributed according to a homogeneous spatial Poisson process of intensity  $\lambda(n)$ . From [10, (2.24) and Thm 3.11],  $\lambda(n) = cn \log n$  with  $c > 1$  is necessary and sufficient for full coverage of  $\mathcal{I}$ . Observe that the constant factor multiplying the  $n \log n$  term for full coverage is  $c > 1$  while for full separability it is  $c = 1$ .

*Remark 2:* If sensors have sensing radius  $r(n) = 1/2n$ , then using the Markov inequality and [10, (3.11)], we can show that to cover at least  $\alpha$ ,  $0 < \alpha < 1$ , length of  $\mathcal{I}$  with probability at least  $\beta$ , the necessary and sufficient conditions on  $\lambda(n)$  are identical to those of  $m(n)$  obtained in statement 3 of Theorem 1. Thus partial coverage and partial separability have identical requirements on the sensor density.

*Remark 3:* The sensing radius  $r(n) = 1/(n+1)$ , does satisfy statement 1 of Theorem 1, but in that case, the  $m(n)$  required will be such that  $m(n) \in \Theta(n^2 \log n)$  for full separability in fixed grid model.

### B. Separability of uniformly distributed target-locations

In this subsection, the target locations  $\mathcal{T}_{\text{rnd}}$  are distributed uniformly in  $\mathcal{I}$ . For notational convenience in this subsection we use  $T_i$  instead of  $T_{(i)}$ . In a realization, target location  $T_i$  may not be separable due to either of the following reasons.

- 1)  $(T_i - T_{i-1})$  and  $(T_{i+1} - T_i)$  are both less than  $r(n)$ , and no sensor can identify  $T_i$ .
- 2) There are no sensors uniquely covering  $T_i$ .

As in the previous subsection, we seek  $r(n)$  and  $m(n)$  to achieve full and partial separability and will account for both these failure conditions. Theorem 2 is the main result of this subsection.

*Theorem 2 (Separability of random target locations):* For the sensing region  $\mathcal{I} = [0, 1]$  and target locations  $\mathcal{T}_{\text{rnd}}$ ,  $m(n)$  sensors are deployed uniformly i.i.d. in  $\mathcal{I}$ .

- 1) For full separability of  $n$  target locations, it is necessary that  $r(n) = 1/(c_n n^2)$  for some  $c_n \rightarrow \infty$ .
- 2) Let  $r(n) = 1/(c_n n^2)$ , for some  $c_n \rightarrow \infty$ . Then the following are true.
  - a) For any  $f_n \rightarrow \infty$  if  $m(n) \geq \frac{1}{2r(n)} \left( \log \left( \frac{1}{2r(n)} \right) + f_n \right) = \left( \frac{n^2 c_n}{2} \right) (2 \log n + \log(c_n/2) + f_n)$ , then  $\Pr(\text{all } n \text{ target locations are separable}) \rightarrow 1$ .
  - b) For any  $f_n \rightarrow \infty$  if  $m(n) \leq \frac{1}{2r(n)} \left( \log \left( \frac{1}{2r(n)} \right) - f_n \right) = \left( \frac{n^2 c_n}{2} \right) (2 \log n + \log(c_n/2) - f_n)$ , then  $\Pr(\text{all } n \text{ target locations are separable}) \rightarrow 0$ .
- 3) For any  $0 < \alpha_1, \beta < 1$ , let  $c_1 := \log(1/(1 - (1 - \alpha_1)(1 - \beta)))$ .
  - a) If  $r(n) \leq \frac{1}{2 \left( \frac{n}{c_1} + 1 \right)}$ , then
$$\Pr(\text{at least } \alpha_1 n \text{ targets are more than } r(n) \text{ away from adjacent neighbors}) \geq \beta.$$
  - b) If  $r(n) > \frac{\log \left( \frac{1}{\alpha_1 \beta} \right)}{2n}$ , then
$$\Pr(\text{at least } \alpha_1 n \text{ targets are more than } r(n) \text{ away from adjacent neighbors}) < \beta.$$
- 4) For a given  $0 < \alpha, \beta < 1$ , choose an  $\alpha_1$  such that  $\alpha < \alpha_1 < 1$ . Let  $c_2 := \log(1/(1 - (1 - \alpha)(1 - \beta)))$ ,  $c_3 := \log(1/(\alpha\beta))$  and  $c_1$  is as defined in statement 3 above. Let  $\theta_1(c_1/(2n)) \leq r(n) \leq \theta_2(c_1/(2n))$ , for any  $\theta_1, \theta_2$  such that  $0 < \theta_1 \leq \theta_2 < 1/(1 + (c_1/n))$ . Choose a finite positive constant  $a$  such that  $a > \max\{1, c_2/(2\theta_1 c_1)\}$ .



- a) If  $m(n) \geq \left(\frac{n}{\theta_1(a-1)c_1}\right) \log\left(1 + \frac{1}{c_2 - 2a\theta_2 c_1}\right)$ , then  
 $\Pr(\geq \alpha n \text{ target locations are separable}) \geq \beta$ .
- b) If  $m(n) < \left(\frac{n}{\theta_2(a-1)c_1} - 1\right) \log\left(\frac{1}{c_3 - a\theta_1 c_1}\right)$ , then  
 $\Pr(\geq \alpha n \text{ target locations are separable}) < \beta$ .

Before the proof of Theorem 2, we first characterize the minimum separation between adjacent target locations. Recall the definition of spacings,  $V_i = T_i - T_{i-1}$ ,  $1 \leq i \leq n$  with  $T_0 := 0$ . Lemma 2 characterizes  $\min_{2 \leq i \leq n} V_i$  and is necessary to prove statement 1 of Theorem 2.

*Lemma 2:* Let  $c_n$  be any sequence such that  $c_n \rightarrow \infty$ . For any sequence  $d_n$ , such that  $0 < d_n < 1/n$ ,  $\Pr(\min_{2 \leq i \leq n} V_i \geq d_n) \rightarrow 1$  if and only if  $d_n = \frac{1}{c_n n^2}$ .

*Proof:* From (2) we have:

$$\Pr\left(\min_{2 \leq i \leq n} V_i > d_n\right) = \Pr(V_2 > d_n, V_3 > d_n, \dots, V_n \geq d_n) = (1 - (n-1)d_n)^n.$$

Upper and lower bounds on the preceding probability using (1) are given below.

$$\exp\left(-\frac{n(n-1)d_n}{1 - (n-1)d_n}\right) \leq \Pr\left(\min_{2 \leq i \leq n} V_i > d_n\right) \leq \exp(-n(n-1)d_n).$$

The ‘if’ part of Lemma 2 is proved as follows, let  $d_n = 1/(c_n n^2)$ , for some  $c_n \rightarrow \infty$ . Then  $\exp\left(-\frac{n(n-1)d_n}{1 - (n-1)d_n}\right)$  and  $\exp(-n(n-1)d_n)$  are asymptotically equal to  $\exp\left(-\frac{1}{(1 + \frac{1}{n-1})\frac{1}{c_n} - \frac{1}{n}}\right)$  and  $\exp\left(-\left(1 - \frac{1}{n}\right)\frac{1}{c_n}\right)$  respectively. Thus  $\Pr(\min_{2 \leq i \leq n} V_i > d_n) \rightarrow 1$ . For the ‘only if’ part, let  $d_n \geq 1/(cn^2)$ , for some real constant  $c > 0$ . Then for any  $n > 1$ ,  $\Pr(\min_{2 \leq i \leq n} V_i > d_n) \leq \exp\left(-\left(1 - \frac{1}{n}\right)\frac{1}{c}\right) = 1 - \epsilon$  where  $\epsilon = \exp\left(-\left(1 - \frac{1}{n}\right)\frac{1}{c}\right) > 0$ . The proof of Lemma 2 is complete. ■

*of Theorem 2:* Observe that  $T_i$  cannot be separated if both  $V_i < r(n)$  and  $V_{i+1} < r(n)$ . Defining  $W_i := V_i + V_{i+1}$ , for  $2 \leq i \leq n-1$ , we see from Section II-A, that it is necessary to have  $\min_{2 \leq i \leq n-1} W_i > 2r(n)$  for separability. Let us now characterize this minimum and prove statement 1 in the following two steps.

- 1) We first prove that for some finite constant  $c > 0$ , if  $r(n) = 1/cn^2$  then  $\Pr(\min W_i \geq 2r(n)) < 1$ . Let  $\mathcal{I} := [0, 1]$  be divided into  $k$  equal sized contiguous intervals, referred to as bins in this proof. Recall that the target locations are chosen uniformly i.i.d. in  $\mathcal{I}$ . The event  $(\min_i W_i \geq \frac{2}{k})$  implies that there exists at most 2 target locations in any 2 consecutive bins, i.e.

$$\left(\min_i W_i \geq \frac{2}{k}\right) \Rightarrow \text{there are at most two target locations in any two consecutive bins.} \quad (6)$$

This is illustrated in Fig. 2. Adjacent target locations could be in the same bin (See  $T_i, T_{i+1}$  in Fig. 2) or adjacent bins (See  $T_j, T_{j+1}$  in Fig. 2). The possible locations of targets  $T_{i-1}, T_{i+2}, T_{j-1}, T_{j+2}$  such that  $\min W_i \geq 2/k$  are shown as shaded regions in Fig. 2. The proof of (6) thus follows.

Number the bins starting at 1 from the left. Let  $Y_i$  be the indicator variable that collectively in bins  $i$  and  $i+1$  there are at most 2 target locations.

$$Y_i = \mathbb{1}(\exists \leq 2 \text{ target locations in bins } \{i, i+1\}).$$

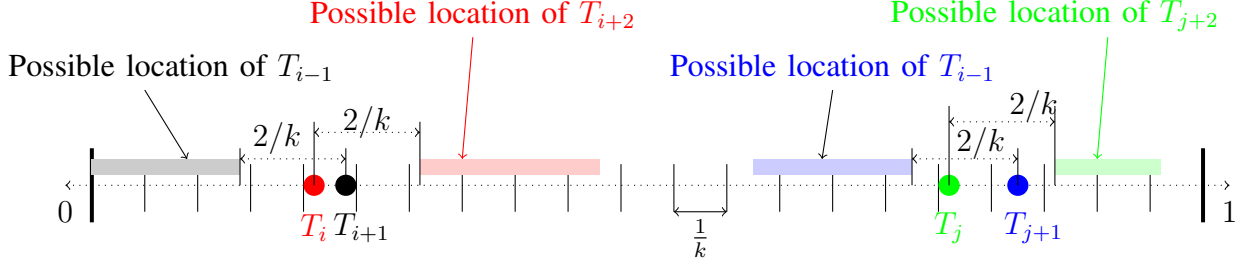


Fig. 2. Illustrating the implication in (6) that  $\min W_i \geq 2/k \Rightarrow$  there are  $\leq 2$  target locations in any two consecutive bins.

Applying the Chernoff bound to a binomial random variable with parameters  $(n, 2/k)$ , the following bound on  $\Pr(Y_i = 1)$  is obtained.

$$\Pr(Y_i = 1) \leq \left(\frac{n}{k}\right)^2 \left(\frac{1 - \frac{2}{k}}{1 - \frac{2}{n}}\right)^{n-2}.$$

Using  $k = cn^2$  in the preceding expression, we have

$$\Pr(Y_i = 1) \leq \left(\frac{1}{cn}\right)^2 \left(1 + \frac{2(cn-1)}{cn(n-2)}\right)^{n-2} \leq \exp\left(2\left(1 - \frac{1}{cn}\right) - 2\log n - 2\log c\right) \rightarrow 0.$$

The second inequality uses  $1 + x < e^x$ . Now using the preceding relation between the events and then the Markov inequality,

$$\Pr\left(\min W_i \geq \frac{2}{cn^2}\right) \leq \Pr\left(\sum_{i=1}^{k-1} Y_i \geq k-1\right) \leq \mathbb{E}(Y_i).$$

Combining this result with the Chernoff bound on  $\Pr(Y_i = 1)$ , we conclude that if  $r(n) = 1/cn^2$  for some finite positive constant  $c$ , then  $\Pr(\min W_i \geq 2r(n)) \rightarrow 0$ .

- 2) From Lemma 2, see that if  $r(n) = 1/(n^2 c_n)$ , for some  $c_n \rightarrow \infty$ , then  $\Pr(\min V_i \geq r(n)) \rightarrow 1$  which implies  $\Pr(\min W_i \geq 2r(n)) \rightarrow 1$ .

This completes the proof of statement 1. We now prove statement 2. Let  $r(n) = 1/(c_n n^2)$ , divide  $\mathcal{I}$  into  $1/(2r(n)) = c_n n^2/2$  intervals of equal width. Every subinterval contains at most one target, with high probability. Recall that for such  $r(n)$  coverage implies identifiability. Then analogous to full separability of targets-on-grid model, it is necessary and sufficient to have at least one sensor in all the subintervals that contain a target. Thus for full separability of uniformly distributed targets, sensors with  $r(n) = \frac{1}{c_n n^2}$ , where  $c_n \rightarrow \infty$ ,  $m(n) = (c_n n^2/2)(\log(n^2 c_n/2) + f_n)$ , for any  $f_n \rightarrow \infty$ , is necessary and sufficient. This completes the proof of statement 2. Remark 4 is the prelude to the proof of statement 3.

*Remark 4:* In partial separability, since at least  $\alpha n$  target locations are identifiable, the number of sensors needed is clearly not sub-linear. Our strategy thus far has been to divide  $\mathcal{I}$  into contiguous non-overlapping cells such that  $r(n)$  is less than half of cell width and choose  $m(n)$  such that there is at least one sensor in each cell. Following this process, there are two approaches.

- 1) Choose a small cell size such that all targets are alone in their cells and then uniquely cover at least  $\alpha n$  of the cells containing targets.
- 2) Choose a large cell size so that at least  $\alpha n$  target locations are alone in their cells and choose  $m(n)$  to uniquely cover all the cells.

We adopt the latter approach in the next proof. Recall that in the targets-on-grid model, the  $m(n)$  required for partial separability is lesser than the full separability case by a factor of  $\log n$ . Thus, it can be expected that the critical number of sensors for partial separability of randomly deployed target locations will be, in the order sense, smaller than  $c_n n^2 \log n$  for any  $c_n \rightarrow \infty$ .

We now prove statement 3. Recall the definition of spacings from Section II-B,  $V_i = T_i - T_{i-1}$ , and define  $Z_i$  as the indicator variable corresponding to the  $i$ -th target location as follows:

$$Z_i := \mathbb{1}(V_i > r(n) \text{ and } V_{i+1} > r(n)).$$

From (3),  $\mathbb{E}(Z_i) = \Pr(Z_i = 1) = (1 - 2r(n))^n$ . Using (4), for any  $\alpha_1$  such that  $\alpha < \alpha_1 < 1$ , we have:

$$1 - \frac{1 - (1 - 2r(n))^n}{1 - \alpha_1} \leq \Pr\left(\sum_{i=1}^n Z_i \geq \alpha_1 n\right) \leq \frac{(1 - 2r(n))^n}{\alpha_1}. \quad (7)$$

We first prove statement 3a. Let  $r(n)$  be chosen such that  $r(n) < 0.5 / ((n/c_1) + 1)$ . Then using the definition of  $c_1$ , the following equivalence is direct.

$$r(n) < \frac{0.5}{\frac{n}{c_1} + 1} \iff \exp\left(-\frac{n}{\frac{1}{2r(n)} - 1}\right) > 1 - (1 - \alpha_1)(1 - \beta). \quad (8)$$

Thus using (7) and (1) in the second inequality of (8), we have

$$\Pr\left(\sum_{i=1}^n Z_i \geq \alpha_1 n\right) \geq 1 - \frac{1 - (1 - 2r(n))^n}{1 - \alpha_1} > \beta.$$

This completes the proof of statement 3a. To prove statement 3b, let  $r(n) > \frac{1}{2n} \log\left(\frac{1}{\alpha_1 \beta}\right)$ , then from (7),  $\Pr(\sum_{i=1}^n Z_i \geq \alpha_1 n) \leq (1 - 2r(n))^n / \alpha_1 < \beta$ . This completes the proof of statement 3.

Before the proof of statement 4, we first obtain bounds on the probability of having at least  $\alpha n$  identifiable target locations as in (10). Towards that, let  $r(n) = c_1 / (4n)$ , choose a constant  $a > 1$ , and define the indicator random variable  $W_i$  as follows.

$$W_i := \mathbb{1}(V_i > ar(n) \text{ \& } V_{i+1} > ar(n) \text{ \& } \exists \text{ at least 1 sensor that uniquely senses the target}).$$

Once again for notational convenience, we use  $m$  instead of  $m(n)$  for the rest of this proof. Since the target locations and sensor locations are chosen independently, we have

$$\begin{aligned} \Pr(W_i = 1) &= \Pr(V_i > ar(n) \text{ \& } V_{i+1} > ar(n)) \Pr(\geq 1 \text{ sensors uniquely sense target } i) \\ &= (1 - 2ar(n))^n (1 - (1 - 2(a-1)r(n))^m). \end{aligned}$$

The bounds on  $\Pr(W_i = 1)$  are obtained using (1) in the preceding expression.

$$\exp\left(-\frac{2anr(n)}{1 - 2ar(n)} - \frac{1}{e^{2(a-1)mr(n)} - 1}\right) \leq \Pr(W_i = 1) \leq \exp\left(-2anr(n) - \exp\left(-\frac{m}{\frac{1}{2(a-1)r(n)} - 1}\right)\right). \quad (9)$$

Using (4) in (9), we have

$$\frac{\Pr(W_i = 1) - \alpha}{1 - \alpha} \leq \Pr\left(\sum_{i=1}^n W_i \geq \alpha n\right) \leq \frac{\sum_{i=1}^n \Pr(W_i = 1)}{\alpha n}. \quad (10)$$

Next, we prove statement 4a. Using  $\theta_1 < 2nr(n)/c_1 < \theta_2$ , for large  $n$ , where  $n > 2a\theta_2c_1$ , we have the first implication.

$$\begin{aligned}
m &\geq \left( \frac{n}{c_1\theta_1(a-1)} \right) \log \left( 1 + \frac{1}{c_2 - 2a\theta_2c_1} \right) \\
&\Rightarrow m \geq \left( \frac{1}{2(a-1)r(n)} \right) \log \left( 1 + \frac{1}{\log \left( \frac{1}{1-(1-\alpha)(1-\beta)} \right) - \frac{2nar(n)}{1-2ar(n)}} \right). \\
&\Leftrightarrow \frac{1}{e^{2m(a-1)r(n)} - 1} \leq \log \left( \frac{1}{1 - (1-\alpha)(1-\beta)} \right) - \frac{2anr(n)}{1-2ar(n)}. \\
&\Leftrightarrow \exp \left( -\frac{2anr(n)}{1-2ar(n)} - \frac{1}{e^{2(a-1)mr(n)} - 1} \right) \geq \alpha + (1-\alpha)\beta.
\end{aligned}$$

The second and third equivalences are obtained by rearranging terms. Thus using the final expression and (9) in (10), we have

$$\Pr \left( \sum_{i=1}^n W_i \geq \alpha n \right) \geq \frac{\exp \left( -\frac{2anr(n)}{1-2ar(n)} - \frac{1}{\exp(2m(a-1)r(n)) - 1} \right) - \alpha}{1 - \alpha} \geq \beta.$$

This completes the proof of statement 4a. Next we give the proof of statement 4b. For  $\theta_1 < 2nr(n)/c_1 < \theta_2$ , we have the following using the definition of  $c_3$ .

$$\begin{aligned}
m &< \left( \frac{n}{\theta_2(a-1)c_1} - 1 \right) \log \left( \frac{1}{c_3 - a\theta_1c_1} \right) \\
&\Rightarrow m < \left( \frac{1}{2(a-1)r(n)} - 1 \right) \log \left( \frac{1}{\log \left( \frac{1}{\alpha\beta} \right) - 2nar(n)} \right). \\
&\Leftrightarrow \exp \left( -\frac{m}{\frac{1}{2(a-1)r(n)} - 1} \right) > \log \left( \frac{1}{\alpha\beta} \right) - 2anr(n).
\end{aligned}$$

Further using the preceding expression and (9) in (10)

$$\Pr \left( \sum_{i=1}^n W_i \geq \alpha n \right) \leq \frac{\exp \left( -2anr(n) - \exp \left( -\frac{m}{\frac{1}{2(a-1)r(n)} - 1} \right) \right)}{\alpha} < \beta.$$

This completes the proof of Theorem 2. ■

*Remark 5:* The following natural schemes for partial separability exist:

- 1) Following the partial separability of targets-on-grid model, choose the sensing radius  $r(n)$  such that  $\alpha n$  target locations are at least  $2r(n)$  away from their adjacent neighbors with probability  $\geq \beta$ , and cover those particular  $\alpha n$  target locations with high probability. This will require  $r(n) = \theta/(2n)$  and  $m(n) = (n/\theta) \log(n/\theta)$ , for some constant  $\theta > 0$ .
- 2) Following the full separability of random targets, choose the sensing radius  $r(n)$  such that all  $n$  target locations are at least  $2r(n)$  away from their adjacent neighbors with high probability and cover  $\alpha n$  of them with probability  $\geq \beta$ . This will require  $r(n) = \theta/(c_n n^2)$  and  $m(n) = \tilde{\theta} c_n n^2$ , for some constants  $\theta, \tilde{\theta}$  and some  $c_n \rightarrow \infty$ .

Observe from the partial separability results that we have  $r(n) \in \Theta(1/n)$  and  $m(n) \in \Theta(n)$ , which are tighter than both the above two approaches.

*Remark 6:* Modeling a fixed number of target locations may seem impractical, so let target locations be realizations of a homogeneous spatial Poisson process of intensity  $n$ , independent of sensor deployment. The results for separability in this Poisson target deployment are similar to the (uniform) random target model. The proof is a special case of the proof of Theorem 4 with  $\gamma = 0$  and is omitted.

#### IV. SEPARABILITY IN 2-DIMENSIONS

In this section, the region of interest is  $\mathcal{I}^2 := [0, 1]^2$ . Each sensor senses all points within  $r(n)$  (circle of radius  $r(n)$ ) from it. Theorem 3 summarizes the results of this section.

*Theorem 3 (Separability in 2-dimensions):* The sensing region is  $\mathcal{I}^2$  and  $m(n)$  denotes the number of sensors that are deployed uniformly i.i.d. in  $\mathcal{I}^2$ .

- 1) In the targets on grid model, the set of  $n$  target locations,  $\mathcal{T}_g^2$ , are the mid points of the cells formed when we tessellate  $\mathcal{I}^2$  into  $n$  square cells, each of size  $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$ . For the targets on grid model, the following are true.
  - a)  $0 < \pi r(n)^2 < (\pi/n)$  is necessary for separability.
  - b) Let  $\pi r(n)^2 = \pi/4n$ , then
    - i) If  $m(n) \geq \frac{4n}{\pi} (\log(\frac{4n}{\pi}) + c_n)$ , for any  $c_n \rightarrow \infty$ , then  $\Pr(\text{all } n \text{ target locations are separable}) \rightarrow 1$ .
    - ii) If  $m(n) \leq \frac{4n}{\pi} (\log(\frac{4n}{\pi}) - c_n)$ , for any  $c_n \rightarrow \infty$ , then  $\Pr(\text{all } n \text{ target locations are separable}) \rightarrow 0$ .
  - c) Let  $\pi r(n)^2 = \pi/4n$ . Given  $0 < \alpha < 1$ , and  $0 < \beta < 1$ , the following are true.
    - i) If  $m(n) \geq (4n/\pi) \log\left(\frac{1}{(1-\alpha)(1-\beta)}\right)$  then  $\Pr(\text{at least } \alpha n \text{ targets are separable}) > \beta$ .
    - ii) If  $m(n) < (4n/\pi - 1) \log\left(\frac{1}{1-\alpha\beta}\right)$  then  $\Pr(\text{at least } \alpha n \text{ targets are separable}) < \beta$ .
- 2) In the random target case, the  $n$  target locations, denoted by  $\mathcal{T}_{\text{rnd}}^2$ , are deployed uniformly i.i.d. in  $\mathcal{I}^2$ . The following are true.
  - a) For full separability of  $n$  target locations, it is necessary that  $r(n)^2 = 1/(c_n n)$  for some  $c_n \rightarrow \infty$ .
  - b) Let  $\pi r(n)^2 = 1/(nc_n)$ , for some  $c_n \rightarrow \infty$ . The following are true
    - i) For any  $g_n \rightarrow \infty$ , if  $m(n) \leq \left(\frac{1}{\pi r(n)^2}\right) \left(\log \frac{1}{\pi r(n)^2} + g_n\right) = (nc_n) (\log nc_n + g_n)$ , then  $\Pr(\text{all } n \text{ target locations are separable}) \rightarrow 1$ .
    - ii) For any  $g_n \rightarrow \infty$ , if  $m(n) \leq \left(\frac{1}{\pi r(n)^2}\right) \left(\log \frac{1}{\pi r(n)^2} - g_n\right) = (nc_n) (\log nc_n - g_n)$ , then  $\Pr(\text{all } n \text{ target locations are separable}) \rightarrow 0$ .
  - c) For any  $0 < \alpha_1, \beta < 1$ , let  $c_1 := \log(1/(1 - (1 - \alpha_1)(1 - \beta)))$  and  $a > 1$  be a finite constant.
    - i) If  $\pi r(n)^2 \leq \frac{1}{a^2 \left(\frac{n-1}{c_1} + 1\right)}$ , then  $\Pr(\text{at least } \alpha_1 n \text{ targets are more than } ar(n) \text{ away from adjacent neighbors}) \geq \beta$ .
    - ii) If  $\pi r(n)^2 > \frac{\log\left(\frac{1}{\alpha_1 \beta}\right)}{a^2 n}$ , then  $\Pr(\text{at least } \alpha_1 n \text{ targets are more than } ar(n) \text{ away from adjacent neighbors}) < \beta$ .

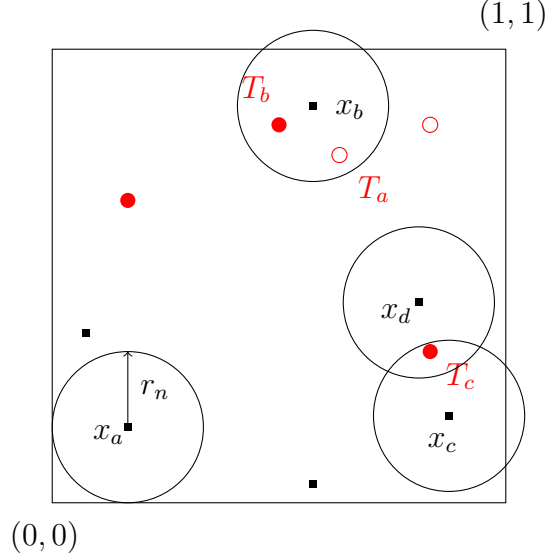


Fig. 3. Illustrating Identifiability in 2 dimensions. The description of targets and sensors are identical to those in Fig. 1

- d) For a given  $0 < \alpha, \beta < 1$ , choose an  $\alpha_1$  such that  $\alpha < \alpha_1 < 1$ . Let  $c_2 := \log(1/(1 - (1 - \alpha)(1 - \beta)))$ ,  $c_3 := \log(1/(\alpha\beta))$  and  $c_1$  is as defined in statement 2c. Let  $\theta_1(c_1/(a^2n)) \leq \pi r(n)^2 \leq \theta_2(c_1/(a^2n))$ , for any  $\theta_1, \theta_2$  such that  $0 < \theta_1 \leq \theta_2 < 1/(1 + (c_1/n))$ . Choose a finite positive constant  $a$  such that  $a^2 > \max\{1, c_2/(2\theta_1 c_1)\}$ .

i) If  $m(n) \geq \left(\frac{n}{\theta_1(a-1)^2 c_1}\right) \log\left(1 + \frac{1}{c_2 - a^2 \theta_2 c_1}\right)$ , then

$\Pr(\geq \alpha n \text{ target locations are separable}) \geq \beta$ .

ii) If  $m(n) < \left(\frac{n}{\theta_2(a-1)^2 c_1} - 1\right) \log\left(\frac{1}{c_3 - a^2 \theta_1 c_1}\right)$ , then

$\Pr(\geq \alpha n \text{ target locations are separable}) < \beta$ .

*Proof:* The proof of statement 1 is self-evident and is omitted. It may be generalized to any  $r(n)$  that satisfies the necessary condition  $0 < \pi r(n)^2 < \pi/n$ .

Next we prove statement 2a. Divide  $[0, 1]^2$  into  $k^2$  equally sized squares, henceforth termed bins. Consider a ‘super-bin’ to be a set of  $2 \times 2$  adjacent bins (i.e., of size  $\frac{2}{k} \times \frac{2}{k}$ ). Similar to the proof of statement 1 of Theorem 2, the following hold.

- 1) If  $k = cn$ , then  $\Pr(\text{each super-bin has } \leq 1 \text{ target location}) \rightarrow 0$ .
- 2) If  $k = c_n n$ , for some  $c_n \rightarrow \infty$ , then  $\Pr(\text{each super-bin has } \leq 1 \text{ target location}) \rightarrow 1$ .

This also ensures that the minimum distance between adjacent targets to be of the form  $1/nc_n$  for some  $c_n \rightarrow \infty$ .

Thus we choose  $\pi r(n)^2 = 1/nc_n$ . Another way to see the minimum distance condition is as follows. Let  $\mathcal{E}_i$  be the event that no other target location is within  $r(n)$  of the  $i$ -th target location. From the Markov inequality, we know that

$$\Pr\left(\sum_{i=1}^n \mathcal{E}_i \geq n\right) \leq \mathbb{E}(\mathcal{E}_i) \leq \exp(-(n-1)\pi r(n)^2).$$

If  $\pi r(n)^2 = c/n$  for some finite positive constant  $c$ , then

$$\Pr \left( \sum_{i=1}^n \mathcal{E}_i \geq n \right) \leq \exp(-c(1 - 1/n)) < 1.$$

Thus we need  $\pi r(n)^2 = 1/nc_n$  for any  $c_n \rightarrow \infty$  to ensure that no two target locations are within  $r(n)$  distance of each other. Thus the proof of statement 2a is complete. The proof of statement 2b is a direct extension of the coupon collector result and is omitted.

Once again for partial separability, the results in statements 2c, 2d are identical to their one dimensional counterparts. The proofs are identically obtained by re-defining  $W_i$  as

$$W_i = \mathbb{1} \text{ (No other target is within } ar(n) \text{ of target } i \text{ \& } \exists \text{ at least 1 sensor within } (a-1)r(n) \text{ of target } i) \text{.}$$

■

## V. SEPARABILITY IN THE PRESENCE OF ADVERSARIAL SENSORS

In this section, we consider the sensing area to be  $\mathcal{I} = [0, 1]$  and sensors are deployed according to a spatial Poisson process of intensity  $m$  on  $\mathcal{I}$ . Let  $M(\sim \text{Poisson}(m))$  be the random variable that denotes the number of sensors. In addition, we assume that a subset,  $\mathcal{A}$ , of the set of sensors  $\mathcal{S} := \{1, 2, \dots, M\}$ , act as adversaries. We also assume that all sensors report binary observations and the sensor locations are known apriori. The sensors in the set  $\mathcal{S} \setminus \mathcal{A}$  report their observations faithfully, we term them “good” sensors. The set  $\mathcal{A}$  of adversarial sensors report an output which may or may not depend on their observation. Each of the  $M$  sensors is an adversary i.i.d. with probability  $\gamma$ ,  $0 < \gamma < 1/2$ , independent of anything else; in other words, the good and adversarial sensors are distributed according to independent spatial Poisson processes of intensity  $(1 - \gamma)m$  and  $\gamma m$  respectively. Note that the set of adversaries,  $\mathcal{A}$ , is unknown to us for the purpose of decoding the target configuration. First consider the targets-on-grid model, recall the following results from Theorem 1. Let the sensing radius be chosen as  $r(n) = a/2n$  (or  $r(n) = (2 - a)/2n$ ) for  $0 < a \leq 1$ . The following results hold.

- 1)  $m(n) \geq (n/a)(\log n + c_n)$  guarantees full separability for any  $c_n \rightarrow \infty$ .
- 2)  $m(n) \geq (n/a) \log(1/[(1 - \alpha)(1 - \beta)])$  guarantees partial separability.

Observe that without any adversaries, the set of sensors that uniquely cover a particular target location have the same observations (either all zero or all one depending on the presence of a target at the location). Adversaries corrupt the set of sensor observations, and thus the set of sensors that uniquely sense target  $i$  give an arbitrary binary vector of observations. We assume that the adversaries don’t have knowledge of the number of sensors that uniquely sense any target. It is easy to see  $0 < r(n) < 1/n$ , similar to statement 1 from Theorem 1, since we need the good sensor observations to decode the target configuration. We show that if any  $0 < \gamma < (1/2)$  fraction of sensors act as adversaries, then, with high probability,  $m \in \Theta(n \log n)$  ensures that  $\mathcal{T}_g$  is full separable. Consider the following sub-optimal scheme to decode the target locations from the set of observations and sensor locations.

- 1) To decode the configuration of target  $i$ , we only use the observations from set of sensors that cover only target  $i$ .
- 2) For  $0 < \gamma < 1/2$ , we will prove that the number of adversarial sensors that uniquely cover target  $t_i$  is dominated by the number of good sensors that uniquely cover target  $t_i$ ,  $\forall 1 \leq i \leq n$ . Thus ‘majority decoding’ on the set of outputs (corresponding to the set

of sensors that uniquely cover target  $t_i$ ) is necessary and sufficient to decode the state of target location  $t_i$  for  $1 \leq i \leq n$  independent of the adversary's behavior.

Let  $A_i, G_i$ , respectively be the random variables that denote number of adversarial sensors and number of good sensors that only sense target location  $i, 1 \leq i \leq n$ . Recall that  $A_i, G_i$  are independent Poisson random variables with intensities  $\lambda_A = \gamma\lambda, \lambda_G = \bar{\gamma}\lambda$  respectively, where  $\bar{\gamma} := 1 - \gamma$  and  $\lambda = m/n$ . Let  $Q_i := G_i - A_i$ . The main result of this section is given in Theorem 4. We will prove Theorem 4 for  $r(n) = 1/2n$ , however the result holds for  $r(n) = a/2n$  (or  $(2-a)/(2n)$ ) for  $0 < a \leq 1$ , with an appropriate change in the constant factors.

*Theorem 4 (Separability of targets-on-grid in the presence of adversaries):* Let  $r(n) = 1/2n$ ,  $0 < \gamma < 0.5$  and the target locations be  $\mathcal{T}_g$ .

- 1) For any  $\epsilon > 0$ , if  $\frac{m}{n \log n} \geq \left( \frac{1+\epsilon}{1-2\sqrt{\gamma\bar{\gamma}}} \right)$ , then  $\Pr(\text{all } n \text{ target locations are separable}) \rightarrow 1$ .
- 2) For any  $0 < \alpha, \beta < 1$ , and any  $\epsilon > 0$ , if  $\frac{m}{n} \geq \left( \frac{1+\epsilon}{1-2\sqrt{\gamma\bar{\gamma}}} \right) \log \left( \frac{1}{(1-\alpha)(1-\beta)} \right)$ , then  $\Pr(\text{at least } \alpha n \text{ target locations are identifiable}) \geq \beta$ .

*Proof:* We will prove statement 1 first. Since  $A_i$  and  $G_i$  are independent Poisson random variables, the moment generating function (MGF) of  $Q_i$  is

$$\text{MGF}_{Q_i}(r) = \exp(-\lambda_G + \lambda_G e^r - \lambda_A + \lambda_A e^{-r}) \quad \forall r \in \mathbb{R}.$$

Since  $\text{MGF}_{-A}(-r) = \text{MGF}_A(r)$  and using the Chernoff bound for  $-Q_i > 0$ , we have

$$\Pr(Q_i < 0) \leq \inf_{r \geq 0} \text{MGF}_{Q_i}(-r) = \exp\left(-\lambda_G - \lambda_A + 2\sqrt{\lambda_G \lambda_A}\right).$$

The complement of this probability gives the required lower bound on  $\Pr(Q_i > 0)$ . Further, use  $\lambda_G = \bar{\gamma}\lambda, \lambda_A = \gamma\lambda$ , where  $0 < \gamma < 1/2$ , in the preceding expression to get

$$\Pr(Q_i > 0) \geq 1 - \exp\left(-\left(1 - 2\sqrt{\gamma\bar{\gamma}}\right)\lambda\right). \quad (11)$$

Note that for  $0 < \gamma < (1/2)$ , the function  $\gamma(1 - \gamma)$  is concave increasing and has a supremum value of  $1/4$  at  $\gamma = 0.5$ . Thus  $1 - 2\sqrt{\gamma\bar{\gamma}} > 0$ , proving that for  $\lim_{n \rightarrow \infty} \Pr(Q_i > 0) = \lim_{\lambda \rightarrow \infty} \Pr(Q_i > 0) = 1$ . Using independence of  $Q_i$  and  $c := 1 - 2\sqrt{\gamma\bar{\gamma}}$ , we see that

$$\Pr(\text{all } n \text{ target locations are separable}) = \prod_{i=1}^n \Pr(Q_i > 0) \geq (1 - e^{-c\lambda})^n.$$

Using the inequality in (1), and  $\lambda = \frac{m}{n} \geq \frac{(1+\epsilon) \log n}{c}$ , we have:

$$\prod_{i=1}^n \Pr(Q_i > 0) \geq \exp\left(-\frac{n}{e^{c\lambda} - 1}\right) \geq \exp\left(-\frac{1}{n^\epsilon - n^{-1}}\right) \rightarrow 1$$

thus proving statement 1. It is easy to see from statement 2b of Theorem 1 that if  $\frac{m}{n} \leq \log n - c_n$  for any  $c_n \rightarrow \infty$ , then  $\Pr(\text{all } n \text{ target locations are separable}) \rightarrow 0$ . Next we prove statement 2. Using (4) and (11) with  $c = 1 - 2\sqrt{\gamma\bar{\gamma}}$ , we see that

$$\begin{aligned} \Pr(\text{at least } \alpha n \text{ target locations are separable}) &= \Pr\left(\sum_{i=1}^n \mathbb{1}(Q_i > 0) \geq \alpha n\right) \\ &\geq \frac{\Pr(Q_i > 0) - \alpha}{1 - \alpha} \geq \frac{1 - \alpha - e^{-c\lambda}}{1 - \alpha}. \end{aligned}$$



Using  $\frac{m}{n} = \lambda > \frac{1}{1-2\sqrt{\gamma\bar{\gamma}}} \log \left( \frac{1}{(1-\alpha)(1-\beta)} \right)$  in the preceding expression, it is easy to see that

$$\Pr(\text{at least } \alpha n \text{ target locations are separable}) > \beta.$$

This completes the proof of statement 2. Using the necessary condition from Theorem 1 and arguing as above, we see that  $m(n) \in \Theta(n)$ . ■

Extending the adversarial setting to the random targets case is identical to the discrete grid setting discussed above and the results are similar to their (no adversaries) ideal binary proximity sensor counterparts, and is hence omitted.

## VI. CONCLUSION

The separability of an asymptotically large number of static target locations with binary proximity sensors has been addressed. Target locations are modeled as a set of deterministic grid points or by realizations of independent and uniform random variables. Sensor locations were static and lack of control in their deployment was modeled by independent and uniform random variables. Order-optimal scaling laws for full and partial separability were derived in this work. For  $n$  target locations, where  $n \rightarrow \infty$ , the number of sensors needed for full and partial separability in the deterministic grid case were  $\Theta(n \log n)$  and  $\Theta(n)$ , respectively. When target locations are obtained from uniform random variables, then the number of sensors needed for full and partial separability were  $\omega(n^2 \log n)$  and  $\Theta(n)$  respectively. Choices for sensing radius, which is a design parameter, in various cases were provided. The conditions for separability in two dimensions were derived. Finally, it was shown that in the presence of adversarial sensors the scaling laws for separability remain unaffected.

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